MIDTERM: VECTOR BUNDLES AND CHARACTERISTIC CLASSES

Date: **19th November 2024**
The Total points is **110** and the maximum you can score is **100** points.

(1) (5+10+10=25 points)
   (a) Define Schubert symbol and the cell associated to it which is used to obtain a cell structure of the Grassmann manifold $G_n(\mathbb{R}^m)$.
   (b) Use Schubert symbols to compute the CW-complex structure of the Grassmann $G_2(\mathbb{R}^4)$ by listing all the cells and their dimensions.
   (c) In the cell decomposition of $G_n(\mathbb{R}^m)$, let σ be a Schubert symbol such that the corresponding cell $e(\sigma)$ is of maximum dimension. Show $e(\sigma)$ is dense in $G_n(\mathbb{R}^m)$.

(2) (5+10=15 points) Define Stiefel-Whitney numbers of a smooth n-dimensional manifold M. Show that the number of distinct Stiefel-Whitney numbers $w_1^{r_1} \dots w_n^{r_n}[M]$ is same as the number of partitions $p(n)$ of n.

(3) (5+15=20 points) Let ξ and η be an $\mathbb{R}^n$-bundle. Define a bundle map $\xi \to \eta$. Let γ^n be the universal bundle on Grassmann manifold $G_n(\mathbb{R}^\infty)$ and suppose there exist a bundle map $\gamma^n \to \xi$ then show that $w_i(\xi)$ is nonzero for $0 \leq i \leq n$.

(4) (6+9=15 points)
   (a) What is an orientation of a finite dimensional $\mathbb{R}$-vector space? Define oriented vector bundle.
   (b) For any rank n vector bundle ξ over a space B, show that $\xi \oplus \xi$ is an orientable vector pundle.

(5) (6+10+9=25 points)
   (a) Define fundamental class and Euler class of an oriented vector bundle of rank n.
   (b) Let γ^n be the universal bundle, show that the Euler class $e(\gamma^n \oplus \gamma^n) \neq 0$.
   (c) If n is odd show that $2e(\gamma^n \oplus \gamma^n) = 0$.

(6) (10 points) Let M be a Riemannian manifold and $f : M \to M \times M$ be the diagonal embedding $x \mapsto (x, x)$. Show that there is a canonical isomorphism between the normal bundle associated to this embedding and the tangent bundle of M.